CHARACTERIZATIONS OF HIGHER DERIVATIONS

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Abstract

Let \( A \) be a unital algebra and \( D = (\delta_i)_{i \in \mathbb{N}} \) be a family of linear mappings from \( A \) into itself such that \( \delta_0 = id_A \). In this paper, we prove that if
\[
\sum_{i+j+k=n} \delta_i(A)\delta_j(B)\delta_k(C) = 0
\]
for any \( A, B, C \in A \) with \( AB = BC = 0 \) and \( \delta_n(I) = 0 \) for all \( n \geq 1 \), then the restriction \( D = (\delta_i)_{i \in \mathbb{N}} \) to the subalgebra \( \mathcal{R} \) generated by all idempotents of \( A \) is a higher derivation. We also show that this kind of mappings is a higher derivation on \( A \) under some conditions.

1. Introduction

Let \( A \) be a unital algebra. Let \( D = (\delta_i)_{i \in \mathbb{N}} \) be a family of linear mappings from \( A \) into itself such that \( \delta_0 = id_A \). \( D \) is called a higher derivation, if \( \delta_n(AB) = \sum_{i+j=n} \delta_i(A)\delta_j(B) \) for each \( n \in \mathbb{N} \) and \( A, B \in A; D \) is called a Jordan higher derivation, if \( \delta_n(A^2) = \sum_{i+j=n} \delta_i(A)\delta_j(A) \) for each \( n \in \mathbb{N} \) and \( A \in A \). Note that \( \delta_1 \)

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is a Jordan derivation, if \( D = (\delta_i)_{i \in \mathbb{N}} \) is a Jordan higher derivation. It is well known that every derivation is a Jordan derivation and the converse in general is not true. In [8], Herstein showed that every Jordan derivation from a 2-torsion free prime ring into itself is a derivation. In [2], Brešar generalized Herstein’s result to 2-torsion free semiprime rings. Likewise, every higher derivation is a Jordan higher derivation and the converse in general is not true. In [6], Ferrero and Haetinger generalized Brešar’s result to the Jordan higher derivations, they showed that every Jordan higher derivation of a 2-torsion free semiprime ring is a higher derivation. For other related results, see [14, 15].

In general, there are two directions in the study of the local actions of derivations of operator algebras. One is the local derivation problem (for example, see [5, 10, 11]). The other is to study the conditions under which derivations of operator algebras can be completely determined by the action on some sets of operators (for example, see [1, 4, 9, 12]). In [3], Brešar study the local actions of derivations. He proved that if \( \delta \) is an additive mapping from a unital ring \( A \) to a unital \( A \)-bimodule \( M \) such that \( A\delta(B)C = 0 \) for all \( AB = BC = 0 \), then the restriction \( \delta \) to the subring \( \mathcal{R} \) generated by all idempotents of \( A \) is a derivation. In [12], Li and Pan showed that under some conditions, this kind of mappings is a generalized derivation on a unital algebra \( A \). In Section 2, we generalize Brešar’s result and Li’s result to the case of higher derivations, respectively.

2. Main Results

In this section, we always assume that \( A \) is a unital algebra.

Let \( D = (\delta_i)_{i \in \mathbb{N}} \) be a family of linear mappings from \( A \) into itself. We say that \( D = (\delta_i)_{i \in \mathbb{N}} \) satisfies the condition (*) if for each \( A \in A \), any idempotents \( P, Q \in A \) and all \( n \in \mathbb{N} \);

\[
\delta_n(PAQ) = \sum_{i+j=n} \delta_i(PA)\delta_j(Q) + \sum_{i+j=n} \delta_i(P)\delta_j(AQ)
\]
\[- \sum_{i+j+k=n} \delta_i(P) \delta_j(A) \delta_k(Q), \]

and

\[\delta_n(I) = 0, \quad \text{for all } n \geq 1.\]

In order to prove our main result, we first show two lemmas.

**Lemma 2.1.** Suppose that \(D = (\delta_i)_{i \in \mathbb{N}}\) is a family of linear mappings from \(A\) into itself satisfying condition \((*)\). Then for any idempotents \(P_1, \ldots, P_m\) in \(A\),

1. \(\delta_n(P_1 \cdots P_m) = \sum_{i+j=n} \delta_i(P_1) \delta_j(P_2 \cdots P_m)\),
2. \(\delta_n(P_1 \cdots P_m) = \sum_{i+j=n} \delta_i(P_1 \cdots P_{m-1}) \delta_j(P_m)\).

**Proof.** We only prove (1), for the proof of (2) is analogous.

When \(m = 1, 2\), by the condition \((*)\), (1) is true. Suppose that if \(m = t\), (1) is true.

For \(m = t + 1\), by the condition \((*)\), it follows that

\[\delta_n(P_1 \cdots P_{t+1}) = \sum_{i+j=n} \delta_i(P_1 \cdots P_t) \delta_j(P_{t+1}) + \sum_{i+j=n} \delta_i(P_1 \cdots P_{t+1}) \delta_j(P_{t+1}) \]

\[- \sum_{i+j+k=n} \delta_i(P_1) \delta_j(P_2 \cdots P_t) \delta_k(P_{t+1}) \]

\[= \sum_{i+j=n} \delta_i(P_1) \delta_j(P_2 \cdots P_{t+1}).\]

This concludes the proof. \(\Box\)

**Lemma 2.2.** Suppose that \(D = (\delta_i)_{i \in \mathbb{N}}\) is a family of linear mappings from \(A\) into itself satisfying condition \((*)\). Then for any idempotents \(P_1, \ldots, P_m, Q_1, \ldots, Q_s\) in \(A\) and every \(A \in A\),

\[\delta_n(I) = 0, \quad \text{for all } n \geq 1.\]
\[ \delta_n(P_1 \cdots P_m AQ_1 \cdots Q_s) \]

\[ = \sum_{i+j=n} \delta_i(P_1 \cdots P_m A) \delta_j(Q_1 \cdots Q_s) + \sum_{i+j=n} \delta_i(P_1 \cdots P_m) \delta_j(AQ_1 \cdots Q_s) \]

\[ - \sum_{i+j+k=n} \delta_i(P_1 \cdots P_m) \delta_j(A) \delta_k(Q_1 \cdots Q_s). \]  

(2.1)

**Proof.** We first show that for any positive integer \( m \),

\[ \delta_n(P_1 \cdots P_m AQ) = \sum_{i+j=n} \delta_i(P_1 \cdots P_m A) \delta_j(Q) + \sum_{i+j=n} \delta_i(P_1 \cdots P_m) \delta_j(AQ) \]

\[ - \sum_{i+j+k=n} \delta_i(P_1 \cdots P_m) \delta_j(A) \delta_k(Q). \]  

(2.2)

If \( m = 1 \), by the condition (*), (2.2) is true. Suppose that if \( m = t \), (2.2) is true.

For \( m = t + 1 \), by the condition (*) and Lemma 2.1, it follows that

\[ \delta_n(P_1 \cdots P_{t+1} AQ) \]

\[ = \sum_{i+j=n} \delta_i(P_1 \cdots P_{t+1} A) \delta_j(Q) + \sum_{i+j=n} \delta_i(P_1) \delta_j(P_2 \cdots P_{t+1} AQ) \]

\[ - \sum_{i+j+k=n} \delta_i(P_1) \delta_j(P_2 \cdots P_{t+1} A) \delta_k(Q) \]

\[ = \sum_{i+j=n} \delta_i(P_1 \cdots P_{t+1} A) \delta_j(Q) + \sum_{i+j+k=n} \delta_i(P_1) \delta_j(P_2 \cdots P_{t+1}) \delta_k(AQ) \]

\[ - \sum_{i+j+k+l=n} \delta_i(P_1) \delta_j(P_2 \cdots P_{t+1}) \delta_k(A) \delta_l(Q) \]

\[ = \sum_{i+j=n} \delta_i(P_1 \cdots P_{t+1} A) \delta_j(Q) + \sum_{i+j=n} \delta_i(P_1 P_2 \cdots P_{t+1}) \delta_j(AQ) \]

\[ - \sum_{i+j+k=n} \delta_i(P_1 P_2 \cdots P_{t+1}) \delta_j(A) \delta_k(Q). \]
Now, we show (2.1) is true.

If \( s = 1 \), by (2.2), (2.1) is true. Suppose that if \( s = t \), (2.1) is true.

For \( s = t + 1 \), by (2.2), the condition (*) and Lemma 2.1, it follows that
\[
\delta_n(P_1 \cdots P_mA Q_l \cdots Q_{t+1})
\]
\[
= \sum_{i+j=n} \delta_i(P_1 \cdots P_m A Q_l \cdots Q_{t+1}) \delta_j(Q_{t+1}) + \sum_{i+j=n} \delta_i(P_1 P_2 \cdots P_m) \delta_j(A Q_l \cdots Q_{t+1})
\]
\[
- \sum_{i+j+k=n} \delta_i(P_1 P_2 \cdots P_m) \delta_j(A Q_l \cdots Q_{t+1}) \delta_k(Q_{t+1})
\]
\[
= \sum_{i+j+k=n} \delta_i(P_1 \cdots P_m A) \delta_j(Q_1 \cdots Q_{t+1}) \delta_k(Q_{t+1})
\]
\[
+ \sum_{i+j=n} \delta_i(P_1 P_2 \cdots P_m) \delta_j(A Q_l \cdots Q_{t+1})
\]
\[
- \sum_{i+j+k+l=n} \delta_i(P_1 \cdots P_m A) \delta_j(A) \delta_k(Q_1 \cdots Q_{t+1}) \delta_l(Q_{t+1})
\]
\[
= \sum_{i+j=n} \delta_i(P_1 \cdots P_m A) \delta_j(Q_1 \cdots Q_{t+1}) + \sum_{i+j=n} \delta_i(P_1 P_2 \cdots P_m) \delta_j(A Q_l \cdots Q_{t+1})
\]
\[
- \sum_{i+j+k=n} \delta_i(P_1 \cdots P_m A) \delta_j(A) \delta_k(Q_1 \cdots Q_{t+1}).
\]

This concludes the proof. \( \square \)

**Theorem 2.3.** Let \( A \) be a unital algebra and \( B \) be the subalgebra generated by all idempotents in \( A \). If \( D = (\delta_i)_{i \in \mathbb{N}} \) is a family of linear mappings from \( A \) into itself such that \( \delta_n(ABC) = \sum_{i+j+k=n} \delta_i(A) \delta_j(B) \delta_k(C) \) for any \( A, B, C \in A \) with \( AB = BC = 0 \) and \( \delta_n(I) = 0 \) for all \( n \geq 1 \), then the restriction of \( D = (\delta_i)_{i \in \mathbb{N}} \) to \( B \) is a higher derivation.
Proof. Let $P$ and $Q$ be two idempotents in $\mathcal{A}$. Since for every $A, I, Q \in A$,

\[
(I - P)PAQ = PAQ(I - Q) = 0,
\]

\[
P(I - P)AQ = (I - P)AQ(I - Q) = 0,
\]

\[
(I - P)PA(I - Q) = PA(I - Q)Q = 0,
\]

\[
P(I - P)A(I - Q) = (I - P)A(I - Q)Q = 0,
\]

we have

\[
\sum_{i+j+k=n} \delta_i(I - P) \delta_j(PAQ) \delta_k(I - Q) = 0,
\]

\[
\sum_{i+j+k=n} \delta_i(P) \delta_j((I - P)AQ) \delta_k(I - Q) = 0,
\]

\[
\sum_{i+j+k=n} \delta_i(I - P) \delta_j(PA(I - Q)) \delta_k(Q) = 0,
\]

\[
\sum_{i+j+k=n} \delta_i(P) \delta_j((I - P)A(I - Q)) \delta_k(Q) = 0.
\]

For convenience, we rewrite these identities as

\[
\delta_n(PAQ) = \sum_{i+j=n} \delta_i(PAQ) \delta_j(Q) + \sum_{i+j=n} \delta_i(P) \delta_j(PAQ)
\]

\[
- \sum_{i+j+k=n} \delta_i(P) \delta_j(PAQ) \delta_k(Q),
\]

\[
- \sum_{i+j=n} \delta_i(P) \delta_j(AQ) = - \sum_{i+j=n} \delta_i(P) \delta_j(PAQ)
\]

\[
- \sum_{i+j+k=n} \delta_i(P) \delta_j(AQ) \delta_k(Q)
\]

\[
+ \sum_{i+j+k=n} \delta_i(P) \delta_j(PAQ) \delta_k(Q),
\]

\[
- \sum_{i+j=n} \delta_i(PA) \delta_j(Q) = - \sum_{i+j=n} \delta_i(PAQ) \delta_j(Q)
\]
\[- \sum_{i+j+k=n} \delta_i(P) \delta_j(PA) \delta_k(Q) + \sum_{i+j+k=n} \delta_i(P) \delta_j(PAQ) \delta_k(Q) + \sum_{i+j+k=n} \delta_i(P) \delta_j(A) \delta_k(Q) - \sum_{i+j+k=n} \delta_i(P) \delta_j(A) \delta_k(Q).\]

Note that the sum of the right-hand sides of these four identities is 0. Therefore, the sum of the left-hand sides must be 0. Hence,
\[
\sum_{i+j+k=n} \delta_i(P) \delta_j(A) \delta_k(Q) = \sum_{i+j=n} \delta_i(P) \delta_j(PA) \delta_k(Q) + \sum_{i+j=n} \delta_i(P) \delta_j(AQ) \delta_k(Q) - \sum_{i+j+k=n} \delta_i(P) \delta_j(A) \delta_k(Q).
\]

By Lemma 2.2, we have for any idempotents \(P_1, \ldots, P_m, Q_1, \ldots, Q_s\) in \(A\) and every \(A \in A\),
\[
\delta_n(P_1 \cdots P_m AQ_1 \cdots Q_s) = \sum_{i+j=n} \delta_i(P_1 \cdots P_m A) \delta_j(Q_1 \cdots Q_s) + \sum_{i+j=n} \delta_i(P_1 \cdots P_m) \delta_j(AQ_1 \cdots Q_s) - \sum_{i+j+k=n} \delta_i(P_1 \cdots P_m) \delta_j(A) \delta_k(Q_1 \cdots Q_s).
\]

Setting \(A = I\) in the above relation, we obtain
\[
\delta_n(P_1 \cdots P_m Q_1 \cdots Q_s) = \sum_{i+j=n} \delta_i(P_1 \cdots P_m) \delta_j(Q_1 \cdots Q_s).
\]

This concludes the proof. \(\square\)
Let $\mathcal{M}$ be an $A$-module and $\mathcal{J}$ be an ideal of $A$. We say that $\mathcal{J}$ is a separating set of $\mathcal{M}$, if for every $m, n \in \mathcal{M}$, $m\mathcal{J} = 0$ implies $m = 0$ and $\mathcal{J}n = 0$ implies $n = 0$.

**Theorem 2.4.** Let $\mathcal{J}$ be a separating set of $A$. Suppose $\mathcal{J}$ is contained in the linear span of the idempotents in $A$. If $D = (\delta_i)_{i \in \mathbb{N}}$ is a family of linear mappings from $A$ into itself satisfying condition (*), then $D = (\delta_i)_{i \in \mathbb{N}}$ is a higher derivation.

**Proof.** When $n = 1$, by [7, Theorem 2.2], we have $\delta_1$ is a derivation. Now we assume that
\[
\delta_m(AB) = \sum_{i+j=m} \delta_i(A)\delta_j(B),
\]
for all $A, B \in A$ and for all $1 \leq m < n$.

Since $\mathcal{J}$ is contained in the linear span of the idempotents in $A$, by the condition (*), it follows that for any $S, T \in \mathcal{J}$,
\[
\delta_n(ST) = \sum_{i+j=n} \delta_i(S)\delta_j(T).
\]

For any $S, T \in \mathcal{J}$ and $A \in A$. Since $\mathcal{J}$ is an ideal of $A$, it follows that
\[
\delta_n(SAT) = \delta_n((SA)T) = \sum_{i+j=n} \delta_i(SA)\delta_j(T). \tag{2.3}
\]

On the other hand, by the condition (*),
\[
\delta_n(SAT) = \sum_{i+j=n} \delta_i(SA)\delta_j(T) + \sum_{i+j=n} \delta_i(S)\delta_j(AT) - \sum_{i+j+k=n} \delta_i(S)\delta_j(A)\delta_k(T). \tag{2.4}
\]

Combining (2.3) and (2.4), we have
\[
0 = \sum_{i+j=n} \delta_i(S)\delta_j(AT) - \sum_{i+j+k=n} \delta_i(S)\delta_j(A)\delta_k(T)
\]
\[ = S \delta_n(AT) - S \sum_{i+j=n} \delta_i(A) \delta_j(T). \]

Since \( \mathcal{J} \) is a separating set of \( \mathcal{A} \), it follows that
\[ \delta_n(AT) = \sum_{i+j=n} \delta_i(A) \delta_j(T). \]

For any \( A, B \in \mathcal{A} \) and for any \( T \in \mathcal{J} \), we have
\[ \delta_n(ABT) = \sum_{i+j=n} \delta_i(A) \delta_j(B) \delta_k(T) = \sum_{i+j+k=n} \delta_i(A) \delta_j(B) \delta_k(T) \]
\[ = \sum_{i+j=n} \delta_i(A) \delta_j(B) T + \sum_{i+j=n \atop j \neq 1} \delta_i(AB) \delta_j(T). \]

On the other hand,
\[ \delta_n(ABT) = \sum_{i+j=n} \delta_i(AB) \delta_j(T) = \delta_n(AB)T + \sum_{i+j=n \atop j \neq 1} \delta_i(AB) \delta_j(T). \]

So, we have
\[ \delta_n(AB)T - \sum_{i+j=n} \delta_i(A) \delta_j(B) T = 0. \]

Since \( \mathcal{J} \) is a separating set of \( \mathcal{A} \), it follows that
\[ \delta_n(AB) = \sum_{i+j=n} \delta_i(A) \delta_j(B). \]

Hence, \( D = (\delta_i)_{i \in \mathbb{N}} \) is a higher derivation. \( \square \)

A linear mapping \( f \) from \( \mathcal{A} \) to \( \mathcal{M} \) is called a left (resp., right) multiplier, if \( f(a) = f(1)a \) (resp., \( f(a) = af(1) \)), for every \( a \in \mathcal{A} \). Clearly, left multipliers are left-annihilator-preserving and right multipliers are right-annihilator-preserving. With certain hypotheses on \( \mathcal{A} \) and \( \mathcal{M} \), multipliers are the only annihilator-preserving maps from \( \mathcal{A} \) to \( \mathcal{M} \) (see [12]). When this happens, we have the following theorem:
Theorem 2.5. Let $A$ be a unital algebra. Suppose that the only linear left-annihilator-preserving maps from $A$ into itself are left multipliers and the only linear right-annihilator-preserving maps from $A$ into itself are right multipliers. If $D = (\delta_i)_{i \in \mathbb{N}}$ is a family of linear mappings from $A$ into itself such that \[ \sum_{i+j+k=n} \delta_i(A)\delta_j(B)\delta_k(C) = 0 \] for any $A, B, C \in A$ with $AB = BC = 0$ and $\delta_n = (I) = 0$ for all $n \geq 1$, then $D = (\delta_i)_{i \in \mathbb{N}}$ is a higher derivation.

Proof. When $n = 1$, $A\delta_1(B)C = 0$ for any $A, B, C \in A$ with $AB = BC = 0$. By [12, Proposition 1.1], we have $\delta_1$ is a derivation. Now we assume that

$$\delta_m(ST) = \sum_{i+j=m} \delta_i(S)\delta_j(T),$$

for all $S, T \in A$ and for all $1 \leq m < n$. Then

$$\sum_{i+j+k=n} \delta_i(A)\delta_j(B)\delta_k(C) = \sum_{i+j+k=n} \delta_i(A)\delta_j(B)\delta_k(C) + \sum_{i+j=n} \delta_i(A)\delta_j(B)C$$

$$= \sum_{i+j=n} \delta_i(A)\delta_j(B)C.$$

So $\sum_{i+j=n} \delta_i(A)\delta_j(B)C = 0$. Fix $A, B \in A$ with $AB = 0$, define a mapping $f$ depending on $A$ and $B$ from $A$ into itself by

$$f(T) = \sum_{i+j=n} \delta_i(A)\delta_j(BT),$$

for any $T \in A$. For any $C, D \in A$ with $CD = 0$, we have $ABC = BCD = 0$. So $f(C)D = 0$. By the hypotheses, $f$ is a left multiplier, that is, $f(S) = f(I)S$ for any $S \in A$. Thus,

$$0 = \sum_{i+j=n} \delta_i(A)\delta_j(BS) - \sum_{i+j=n} \delta_i(A)\delta_j(B)S$$
\[
\begin{align*}
&= A\delta_n(BS) + \sum_{i+j+k=n, i \geq 1} \delta_i(A) \delta_j(B) \delta_k(S) - \sum_{i+j=n} \delta_i(A) \delta_j(B)S \\
&= A\delta_n(BS) + \sum_{i+j+k=n, i \geq 1, k \geq 1} \delta_i(A) \delta_j(B) \delta_k(S) + \sum_{i+j=n} \delta_i(A) \delta_j(B)S \\
&\quad - \sum_{i+j=n} \delta_i(A) \delta_j(B)S \\
&= A\delta_n(BS) - A \sum_{i+j=n} \delta_i(B) \delta_j(S) - A\delta_n(BS) \\
&= A\delta_n(BS) - A \sum_{i+j=n} \delta_i(B) \delta_j(S).
\end{align*}
\]

Let
\[
g(T) = \delta_n(TS) - \sum_{i+j=n} \delta_i(T) \delta_j(S).
\]

Then \(Ag(B) = 0\). By the hypotheses, \(g\) is a right multiplier, that is, 
\(g(T) = Tg(1)\) for any \(T \in A\). Since \(g(I) = 0\), we have \(g(T) = 0\). Thus,
\[
\delta_n(TS) = \sum_{i+j=n} \delta_i(T) \delta_j(S).
\]

Hence, \(D = (\delta_i)_{i \in \mathbb{N}}\) is a higher derivation. \(\square\)

References


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